Exercise 6C

1 a $\mathbf{A}=\begin{pmatrix}1&0&0\\0&2&1\\0&1&2\end{pmatrix}$

Step 1:

$$\det(\mathbf{A})=\begin{vmatrix}1&0&0\\0&2&1\\0&1&2\end{vmatrix}$$

$$=1\begin{vmatrix}2&1\\1&2\end{vmatrix}-0\begin{vmatrix}0&1\\0&2\end{vmatrix}+0\begin{vmatrix}0&2\\0&1\end{vmatrix}$$

$$=4-1$$

$$=3$$

Step 2:

$$\mathbf{M}=\begin{Vmatrix}\begin{vmatrix}2&1\\1&2\end{vmatrix}&\begin{vmatrix}0&1\\0&2\end{vmatrix}&\begin{vmatrix}0&2\\0&1\end{vmatrix}\\\begin{vmatrix}0&0\\1&2\end{vmatrix}&\begin{vmatrix}1&0\\0&2\end{vmatrix}&\begin{vmatrix}1&0\\0&1\end{vmatrix}\\\begin{vmatrix}0&0\\2&1\end{vmatrix}&\begin{vmatrix}1&0\\0&1\end{vmatrix}&\begin{vmatrix}1&0\\0&2\end{vmatrix}\end{Vmatrix}$$

$$=\begin{pmatrix}4-1&0-0&0-0\\0-0&2-0&1-0\\0-0&1-0&2-0\end{pmatrix}$$

$$=\begin{pmatrix}3&0&0\\0&2&1\\0&1&2\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}3&0&0\\0&2&-1\\0&-1&2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}3&0&0\\0&2&-1\\0&-1&2\end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

1 b $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Step 1:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$

$$= 6$$

Step 2:

$$\mathbf{M} = \begin{vmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{vmatrix}$$

$$= \begin{pmatrix} 6-0 & 0-0 & 0-0 \\ 0-0 & 3-0 & 0-0 \\ 0-0 & 0-0 & 2-0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Step 3:

$$\mathbf{C} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

1 c $\mathbf{A}=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{3}{5} & -\frac{4}{5}\\ 0 & \frac{4}{5} & \frac{3}{5}\end{pmatrix}$

Step 1:

$$\det(\mathbf{A})=\begin{vmatrix}1 & 0 & 0\\ 0 & \frac{3}{5} & -\frac{4}{5}\\ 0 & \frac{4}{5} & \frac{3}{5}\end{vmatrix}$$

$$=1\begin{vmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{vmatrix}-0\begin{vmatrix}0 & -\frac{4}{5}\\ 0 & \frac{3}{5}\end{vmatrix}+0\begin{vmatrix}0 & \frac{3}{5}\\ 0 & \frac{4}{5}\end{vmatrix}$$

$$=\frac{9}{25}+\frac{16}{25}$$

$$=1$$

Step 2:

$$\mathbf{M}=\begin{vmatrix}\begin{vmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{vmatrix} & \begin{vmatrix}0 & -\frac{4}{5}\\ 0 & \frac{3}{5}\end{vmatrix} & \begin{vmatrix}0 & \frac{3}{5}\\ 0 & \frac{4}{5}\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ \frac{4}{5} & \frac{3}{5}\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & \frac{3}{5}\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & \frac{4}{5}\end{vmatrix}\\ \begin{vmatrix}0 & 0\\ \frac{3}{5} & -\frac{4}{5}\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & -\frac{4}{5}\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & \frac{3}{5}\end{vmatrix}\end{vmatrix}$$

$$=\begin{pmatrix}\frac{9}{25}+\frac{16}{25} & 0-0 & 0-0\\ 0-0 & \frac{3}{5}-0 & \frac{4}{5}-0\\ 0-0 & -\frac{4}{5}-0 & \frac{3}{5}-0\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{3}{5} & \frac{4}{5}\\ 0 & -\frac{4}{5} & \frac{3}{5}\end{pmatrix}$$

Step 3:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

2 a $\mathbf{A}=\begin{pmatrix}1 & -3 & 2\\ 0 & -2 & 1\\ 3 & 0 & 2\end{pmatrix}$

Step 1:

$$\det(\mathbf{A})=\begin{vmatrix}1 & -3 & 2\\ 0 & -2 & 1\\ 3 & 0 & 2\end{vmatrix}$$

$$=1\begin{vmatrix}-2 & 1\\ 0 & 2\end{vmatrix}+3\begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix}+2\begin{vmatrix}0 & -2\\ 3 & 0\end{vmatrix}$$

$$=(-4-0)+3(0-3)+2(0+6)$$

$$=-4-9+12$$

$$=-1$$

Step 2:

$$\mathbf{M}=\begin{Vmatrix}\begin{vmatrix}-2 & 1\\ 0 & 2\end{vmatrix} & \begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}0 & -2\\ 3 & 0\end{vmatrix}\\ \begin{vmatrix}-3 & 2\\ 0 & 2\end{vmatrix} & \begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix} & \begin{vmatrix}1 & -3\\ 3 & 0\end{vmatrix}\\ \begin{vmatrix}-3 & 2\\ -2 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & -3\\ 0 & -2\end{vmatrix}\end{Vmatrix}$$

$$=\begin{pmatrix}-4-0 & 0-3 & 0+6\\ -6-0 & 2-6 & 0+9\\ -3+4 & 1-0 & -2-0\end{pmatrix}$$

$$=\begin{pmatrix}-4 & -3 & 6\\ -6 & -4 & 9\\ 1 & 1 & -2\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}-4 & 3 & 6\\ 6 & -4 & -9\\ 1 & -1 & -2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-4 & 6 & 1\\ 3 & -4 & -1\\ 6 & -9 & -2\end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{-1}\begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -6 & -1 \\ -3 & 4 & 1 \\ -6 & 9 & 2 \end{pmatrix}$$

2 b $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

Step 1:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 2(-2-1) - 3(3-2) + 2(3+4)$$

$$= -6 - 3 + 14$$

$$= 5$$

Step 2:

$$\mathbf{M} = \begin{Vmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \end{Vmatrix}$$

$$= \begin{pmatrix} -2-1 & 3-2 & 3+4 \\ 3-2 & 2-4 & 2-6 \\ 3+4 & 2-6 & -4-9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 7 \\ 1 & -2 & -4 \\ 7 & -4 & -13 \end{pmatrix}$$

Step 3:

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & -\frac{13}{5} \end{pmatrix}$$

2 c $\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

Step 1:

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} - 7\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$$

$$= 3(6-2) - 2(-2-0) - 7(2-0)$$

$$= 12 + 4 - 14$$

$$= 2$$

Step 2:

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 6-2 & -2-0 & 2-0 \\ -4+14 & -6-0 & 6-0 \\ 2-21 & 3+7 & -9-2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

Step 3:

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$$

3 a $\mathbf{A}=\begin{pmatrix}1&0&1\\0&1&0\\2&0&1\end{pmatrix}$

Step 1:

$$\det(\mathbf{A})=\begin{vmatrix}1&0&1\\0&1&0\\2&0&1\end{vmatrix}$$
$$=1\begin{vmatrix}1&0\\0&1\end{vmatrix}-0\begin{vmatrix}0&0\\2&1\end{vmatrix}+1\begin{vmatrix}0&1\\2&0\end{vmatrix}$$
$$=1(1-0)-0(0-0)+1(0-2)$$
$$=-1$$

Step 2:

$$\mathbf{M}_\mathbf{A}=\begin{pmatrix}\begin{vmatrix}1&0\\0&1\end{vmatrix}&\begin{vmatrix}0&0\\2&1\end{vmatrix}&\begin{vmatrix}0&1\\2&0\end{vmatrix}\\\begin{vmatrix}0&1\\0&1\end{vmatrix}&\begin{vmatrix}1&1\\2&1\end{vmatrix}&\begin{vmatrix}1&0\\2&0\end{vmatrix}\\\begin{vmatrix}0&1\\1&0\end{vmatrix}&\begin{vmatrix}1&1\\0&0\end{vmatrix}&\begin{vmatrix}1&0\\0&1\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}1-0&0-0&0-2\\0-0&1-2&0-0\\0-1&0-0&1-0\end{pmatrix}$$
$$=\begin{pmatrix}1&0&-2\\0&-1&0\\-1&0&1\end{pmatrix}$$

Step 3:

$$\mathbf{C}_\mathbf{A}=\begin{pmatrix}1&0&-2\\0&-1&0\\-1&0&1\end{pmatrix}$$

Step 4:

$$\mathbf{C}_\mathbf{A}^\mathrm{T}=\begin{pmatrix}1&0&-1\\0&-1&0\\-2&0&1\end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}_\mathbf{A}^\mathrm{T}$$
$$=\frac{1}{-1}\begin{pmatrix}1&0&-1\\0&-1&0\\-2&0&1\end{pmatrix}$$
$$=\begin{pmatrix}-1&0&1\\0&1&0\\2&0&-1\end{pmatrix}$$

3 b $\mathbf{B}=\begin{pmatrix}2 & 1 & -1\\ 1 & 0 & 1\\ 1 & 2 & 1\end{pmatrix}$

Step 1:

$$\det(\mathbf{B})=\begin{vmatrix}2 & 1 & -1\\ 1 & 0 & 1\\ 1 & 2 & 1\end{vmatrix}$$

$$=2\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix}-1\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}-1\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}$$

$$=2(0-2)-1(1-1)-1(2-0)$$

$$=-4-2$$

$$=-6$$

Step 2:

$$\mathbf{M_B}=\begin{pmatrix}\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}\\ \begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}\\ \begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix} & \begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 0\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}0-2 & 1-1 & 2-0\\ 1+2 & 2+1 & 4-1\\ 1-0 & 2+1 & 0-1\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 0 & 2\\ 3 & 3 & 3\\ 1 & 3 & -1\end{pmatrix}$$

Step 3:

$$\mathbf{C_B}=\begin{pmatrix}-2 & 0 & 2\\ -3 & 3 & -3\\ 1 & -3 & -1\end{pmatrix}$$

Step 4:

$$\mathbf{C_B^T}=\begin{pmatrix}-2 & -3 & 1\\ 0 & 3 & -3\\ 2 & -3 & -1\end{pmatrix}$$

Step 5:

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\mathbf{C}_{\mathbf{B}}^{\mathrm{T}}$$

$$= \frac{1}{-6}\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

3 c $(\mathbf{AB})^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}+0-\frac{1}{3} & 0+\frac{1}{2}+0 & \frac{1}{3}+0+\frac{1}{6} \\ 0+0+1 & 0-\frac{1}{2}+0 & 0+0-\frac{1}{2} \\ \frac{1}{3}+0+\frac{1}{3} & 0+\frac{1}{2}+0 & -\frac{1}{3}+0-\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Therefore, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ as required

4 a $\mathbf{A}=\begin{pmatrix}2&0&3\\k&1&1\\1&1&4\end{pmatrix}$

$$\det(\mathbf{A})=\begin{vmatrix}2&0&3\\k&1&1\\1&1&4\end{vmatrix}$$
$$=2\begin{vmatrix}1&1\\1&4\end{vmatrix}-0\begin{vmatrix}k&1\\1&4\end{vmatrix}+3\begin{vmatrix}k&1\\1&1\end{vmatrix}$$
$$=2(4-1)-0(4k-1)+3(k-1)$$
$$=6+3k-3$$
$$=3(k+1)\ \text{ as required}$$

b

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}1&1\\1&4\end{vmatrix}&\begin{vmatrix}k&1\\1&4\end{vmatrix}&\begin{vmatrix}k&1\\1&1\end{vmatrix}\\\begin{vmatrix}0&3\\1&4\end{vmatrix}&\begin{vmatrix}2&3\\1&4\end{vmatrix}&\begin{vmatrix}2&0\\1&1\end{vmatrix}\\\begin{vmatrix}0&3\\1&1\end{vmatrix}&\begin{vmatrix}2&3\\k&1\end{vmatrix}&\begin{vmatrix}2&0\\k&1\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}4-1&4k-1&k-1\\0-3&8-3&2-0\\0-3&2-3k&2-0\end{pmatrix}$$
$$=\begin{pmatrix}3&4k-1&k-1\\-3&5&2\\-3&2-3k&2\end{pmatrix}$$
$$\mathbf{C}=\begin{pmatrix}3&1-4k&k-1\\3&5&-2\\-3&3k-2&2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}3&3&-3\\1-4k&5&3k-2\\k-1&-2&2\end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$
$$=\frac{1}{3(k+1)}\begin{pmatrix}3&3&-3\\1-4k&5&3k-2\\k-1&-2&2\end{pmatrix}$$

5 $\mathbf{A} = \mathbf{A}^{-1} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$

$$\mathbf{AA}^{-1} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}\begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$$

$$= \begin{pmatrix} 25+ab+8 & 5a-7a-8 & 20+8a+4c \\ 5b-7b+16 & ab+49-16 & 4b-56+8c \\ 10-2b+2c & 2a+14-2c & 8-16+c^2 \end{pmatrix}$$

Since $\mathbf{AA}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$25+ab+8=1$ **(1)**
$5a-7a-8=0$ **(2)**
$20+8a+4c=0$ **(3)**

From **(2)**
$-2a=8 \Rightarrow a=-4$
Substituting $a=-4$ into **(1)** gives:
$25-4b+8=1$
$33-4b=1 \Rightarrow b=8$
Substituting $a=-4$ into **(3)** gives:
$20+8(-4)+4c=0$
$-12+4c=0 \Rightarrow c=3$

6 a $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$

$$\mathbf{A}^2 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-4-3 & -2+3+3 & 2+0+1 \\ 8-12+0 & -4+9+0 & 4+0+0 \\ -6+12-3 & 3-9+3 & -3+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

$$\mathbf{A}^3 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -6+4+3 & 8-5-3 & 6-4-2 \\ -12+12+0 & 16-15+0 & 12-12+0 \\ 9-12+3 & -12+15-3 & -9+12-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\mathbf{A}^3 = \mathbf{I}$ as required

b $\mathbf{A}\mathbf{A}^2 = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Therefore:

$\mathbf{A}^2 = \mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$$

7 a $\mathbf{A}=\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}$

$$\mathbf{A}^2=\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}1+3+0 & 1-3+0 & 0+1+0\\ 3-9+0 & 3+9+3 & 0-3+2\\ 0+9+0 & 0-9+6 & 0+3+4\end{pmatrix}$$

$$=\begin{pmatrix}4 & -2 & 1\\ -6 & 15 & -1\\ 9 & -3 & 7\end{pmatrix}$$

$$\mathbf{A}^3=\begin{pmatrix}4 & -2 & 1\\ -6 & 15 & -1\\ 9 & -3 & 7\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}4-6+0 & 4+6+3 & 0-2+2\\ -6+45+0 & -6-45-3 & 0+15-2\\ 9-9+0 & 9+9+21 & 0-3+14\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 13 & 0\\ 39 & -54 & 13\\ 0 & 39 & 11\end{pmatrix}$$

$\mathbf{A}^3=13\mathbf{A}-15\mathbf{I}$

$$=13\begin{pmatrix}1 & 1 & 0\\ 3 & -3 & 1\\ 0 & 3 & 2\end{pmatrix}-15\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}13 & 13 & 0\\ 39 & -39 & 13\\ 0 & 39 & 26\end{pmatrix}-\begin{pmatrix}15 & 0 & 0\\ 0 & 15 & 0\\ 0 & 0 & 15\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 13 & 0\\ 39 & -54 & 13\\ 0 & 39 & 11\end{pmatrix}$$

Therefore, $\mathbf{A}^3=13\mathbf{A}-15\mathbf{I}$ as required

b $\mathbf{A}^3=13\mathbf{A}-15\mathbf{I}$

Multiplying by $\mathbf{A}^{-1}$ gives:

$\mathbf{A}^3\mathbf{A}^{-1}=13\mathbf{A}\mathbf{A}^{-1}-15\mathbf{I}\mathbf{A}^{-1}$

$\mathbf{A}^2=13\mathbf{I}-15\mathbf{A}^{-1}$

$15\mathbf{A}^{-1}=13\mathbf{I}-\mathbf{A}^2$ as required

7 c $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$

$$= 13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 1 \\ -6 & 15 & -1 \\ 9 & -3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{15}\begin{pmatrix} 9 & 2 & -1 \\ 6 & -2 & 1 \\ -9 & 3 & 6 \end{pmatrix}$$

8 a $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}$

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} + 0\begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix}$$

$$= 2(-12+6) + 0(-8-0) + 1(12-0)$$

$$= -12 + 12$$

$$= 0$$

Therefore, $\mathbf{A}$ is singular

8 b

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & -2 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -12+6 & -16-0 & 12-0 \\ 0-3 & -8-0 & 6-0 \\ 0-3 & -4-4 & 6-0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -16 & 12 \\ -3 & -8 & 6 \\ -3 & -8 & 6 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -6 & 16 & 12 \\ 3 & -8 & -6 \\ -3 & 8 & 6 \end{pmatrix}$$

c $$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$$

$$\mathbf{AC}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -12+0+12 & 6+0-6 & -6+0+6 \\ -24+48-24 & 12-24+12 & -12+24-12 \\ 0+48-48 & 0-24+24 & 0+24-24 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\mathbf{AC}^{\mathrm{T}} = \mathbf{0}$ as required.